PAL
10/11/07

Unified Engineering Problem Set
Week 7 Fall, 2007

SOLUTIONS

M7.1

for all cases, recall rules for tensorial/indicial notation

- Latin subscripts take on values 1, 2, 3
- Greek subscripts take on values 1, 2
- when subscript is repeated in one term, it is a "dummy index" and is summed on
- when subscript appears only once on left side of equation in one term, it is a "free index" and represents separate equations

So .....

(a) $C_{mn} = R_{mnjk} \gamma_j z_k$ (for $m = 1$, $n = 3$)

$\Rightarrow C_{13} = C_{13jk} \gamma_j z_k$

- j and k are dummy indices and are summed on from 1 to 3 (they are latin subscripts)

$\Rightarrow$ same as: $$C_{13} = \sum_{j=1}^{3} \sum_{k=1}^{3} C_{13jk} \gamma_j z_k$$

Thus:

$$\boxed{\begin{aligned} C_{13} = {} & C_{1311}\gamma_1 z_1 + C_{1312}\gamma_1 z_2 + C_{1313}\gamma_1 z_3 \\ & + C_{1321}\gamma_2 z_1 + C_{1322}\gamma_2 z_2 + C_{1323}\gamma_2 z_3 \\ & + C_{1331}\gamma_3 z_1 + C_{1332}\gamma_3 z_2 + C_{1333}\gamma_3 z_3 \end{aligned}}$$

(b) $E = \frac{1}{2}\sigma_{\alpha\beta}\epsilon_{\alpha\beta}$

- α and β are <u>dummy</u> <u>indices</u> and are summed on from 1 to 2 (they are greek subscripts)

$\Rightarrow$ same as: $$\epsilon = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sigma_{\alpha\beta}\epsilon_{\alpha\beta}$$

Thus:

$$\boxed{\epsilon = \frac{1}{2}\left\{\sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{21}\epsilon_{21} + \sigma_{22}\epsilon_{22}\right\}}$$

(c) $H_i = D_{\alpha\beta} P_{\alpha\beta} n_i$

- α and β are repeated in one term and thus are <u>dummy</u> <u>indices</u> and are summed on from 1 to ~~3~~ 2 (they are greek subscripts)

- i is a <u>free index</u> and indicates separate equation (3 as it is a latin subscript)

$\Rightarrow$ same as:

$$H_i = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} b_{\alpha\beta} P_{\alpha\beta} n_i$$

Thus:

$$\boxed{\begin{aligned} (i=1) \quad H_1 &= (b_{11} P_{11} + b_{12} P_{12} + b_{21} P_{21} + b_{22} P_{22}) n_1 \\ (i=2) \quad H_2 &= (b_{11} P_{11} + b_{12} P_{12} + b_{21} P_{21} + b_{22} P_{22}) n_2 \\ (i=3) \quad H_3 &= (b_{11} P_{11} + b_{12} P_{12} + b_{21} P_{21} + b_{22} P_{22}) n_3 \end{aligned}}$$

(d) $\sigma_{31} = l_{3m'} l_{1n'} \sigma'_{m'n'}$

- m' and n' are repeated in one term and are <u>dummy indices</u> are summed on from 1 to 3 (they are latin subscripts)

$\Rightarrow$ same as: $\sigma_{31} = \sum_{m'=1}^{3} \sum_{n'=1}^{3} l_{3m'} l_{1n'} \sigma_{m'n'}$

Thus:

$$\boxed{\begin{aligned} \sigma_{31} = l_{31'} &(l_{11'} \sigma_{1'1'} + l_{12'} \sigma_{1'2'} + l_{13'} \sigma_{1'3'}) \\ + l_{32'} &(l_{11'} \sigma_{2'1'} + l_{12'} \sigma_{2'2'} + l_{13'} \sigma_{2'3'}) \\ + l_{33'} &(l_{11'} \sigma_{3'1'} + l_{12'} \sigma_{3'2'} + l_{13'} \sigma_{3'3'}) \end{aligned}}$$

(e) $f_{pq}(\partial g_q/\partial t) + x_p = 0$

- q is repeated in the first term and is thus a <u>dummy index</u> and is summed on from 1 to 3 (it is a latin subscript)
- p is a <u>free index</u> (not repeated in terms) and indicates there are 3 equations (latin subscript $\Rightarrow$ 1, 2, 3)

$\Rightarrow$ same as: $\displaystyle\sum_{q=1}^{3} f_{pq}\left(\partial g_q/\partial t\right) + x_p = 0$

Thus:

$$\boxed{\begin{aligned}
p=1:&\quad f_{11}\,\partial g_1/\partial t + f_{12}\,\partial g_2/\partial t + f_{13}\,\partial g_3/\partial t + x_1 = 0\\
p=2:&\quad f_{21}\,\partial g_1/\partial t + f_{22}\,\partial g_2/\partial t + f_{23}\,\partial g_3/\partial t + x_2 = 0\\
p=3:&\quad f_{31}\,\partial g_1/\partial t + f_{32}\,\partial g_2/\partial t + f_{33}\,\partial g_3/\partial t + x_3 = 0
\end{aligned}}$$

M7.2

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} \alpha_{111} & \alpha_{221} & 2\alpha_{121} \\ \alpha_{112} & \alpha_{222} & 2\alpha_{122} \\ \alpha_{113} & \alpha_{223} & 2\alpha_{123} \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{Bmatrix}$$

First write out in full (as it may help):

$$M_1 = \alpha_{111} T_{11} + \alpha_{221} T_{22} + 2\alpha_{121} T_{12}$$
$$M_2 = \alpha_{112} T_{11} + \alpha_{222} T_{22} + 2\alpha_{122} T_{12}$$
$$M_3 = \alpha_{113} T_{11} + \alpha_{223} T_{22} + 2\alpha_{123} T_{12}$$

Look at/consider this piece by piece:

(1) The subscript on M must be a free index because it changes with the equation and represents separate equations. It must be latin since it takes on the values 1, 2, 3 ....
(= M_n)

(2) The subscripts on T take on the values 1 and 2 and therefore must be greek. They change independently and thus must be different....

$$(T_{\gamma\beta})$$

(3) The third subscript on α matches the subscript on M....

$$(M_n = \alpha_{??n} T_{\gamma\beta})$$

(4) The second and first subscripts on α match those on T. By making them the same, they are also summed on (as occurs in the equations)

$$\Rightarrow \boxed{M_n = \alpha_{\gamma\beta n} T_{\gamma\beta}}$$

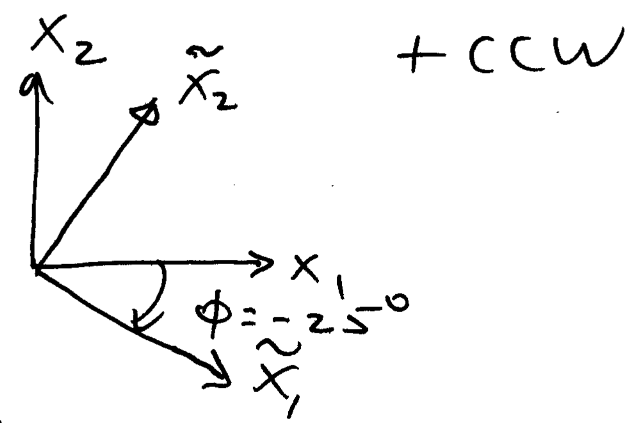
__But__, one must also make the assumption that $T_{\gamma\beta}$ is symmetric ($T_{\gamma\beta} = T_{\beta\gamma}$) and $\alpha_{\gamma\beta n}$ is symmetric in the first two indices ($\alpha_{\gamma\beta n} = \alpha_{\beta\gamma n}$) to get the factor of 2 in the final equations on the T_{12} terms with α_{12n} as multipliers.

M7.3 $\underline{r} = 4\underline{i}_1 - 3\underline{i}_2 - 5\underline{i}_3$

(note: unitless)

(a) Rotation is in $x_1 - x_2$ plane about x_3 axis by angle of $-25°$.

Draw this:



with x_3 out of the paper.

~ represents the rotated system.

$$\tilde{x}_3 = x_3$$

Know that the rotation can be represented via the direction cosines:

$$\tilde{r}_{\tilde{i}} = l_{\tilde{i}j} r_j$$

Determine the direction cosines:

$$l_{\tilde{1}1} = \cos(\phi) = \cos(-25^\circ) = 0.906$$

$$l_{\tilde{1}2} = \cos(-90+\phi) = \cos(90-\phi) = \sin\phi = \sin(-25^\circ) = -0.423$$

$$l_{\tilde{1}3} = \cos(-90^\circ) = 0$$

$$l_{\tilde{2}1} = \cos(90+\phi) = -\sin\phi = 0.423$$

$$l_{\tilde{2}2} = \cos(\phi) = \cos(-25^\circ) = 0.906$$

$$l_{\tilde{2}3} = \cos(90^\circ) = 0$$

$$l_{\tilde{3}1} = \cos(+90^\circ) = 0$$

$$l_{\tilde{3}2} = \cos(+90^\circ) = 0$$

$$l_{\tilde{3}3} = \cos(0^\circ) = 1$$

writing out the rotational equation:

$$\tilde{r}_{\tilde{i}} = l_{\tilde{i}j} r_j$$

$$\Rightarrow \quad \tilde{r}_1 = l_{\tilde{1}1} r_1 + l_{\tilde{1}2} r_2 + l_{\tilde{1}3} r_3$$

$$= (0.906)(4) + (-0.423)(-3) = 4.893$$

$$\tilde{r}_2 = l_{\tilde{2}1} r_1 + l_{\tilde{2}2} r_2 + l_{\tilde{2}3} r_3$$

$$= (0.423)(4) + (0.906)(-3) = -1.026$$

$$\tilde{r}_3 = l_{\tilde{3}1} r_1 + l_{\tilde{3}2} r_2 + l_{\tilde{3}3} r_3$$

$$= r_3 = -5$$

$$\boxed{\tilde{r} = 4.893\,\tilde{i}_1 - 1.026\,\tilde{i}_2 - 5\,\tilde{i}_3}$$

(b) To prove that these expressions are equivalent, one can look at the overall magnitude of the vector (along with directions)

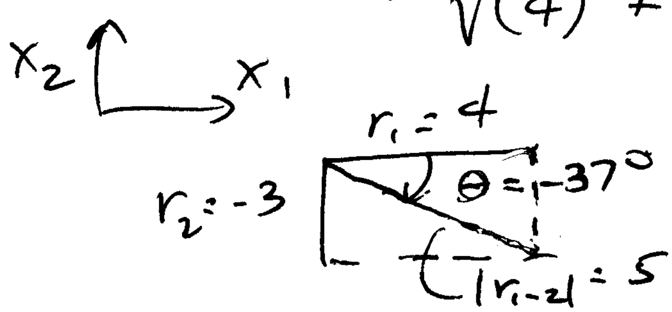
<u>Step 1</u> Note that the magnitude in the 3-direction is the same since there is no rotation there (that axis is rotated about)

<u>Step 2</u> The 2-D vector in the $x_1 - x_2$ plane creates a right triangle with the x_2-magnitude as one side and the x_1-magnitude as the other.

With this, determine the overall magnitude of the vector via:

$$|r_{1-2}| = \sqrt{(r_1)^2 + (r_2)^2}$$

$$= \sqrt{(4)^2 + (-3)^2} = \sqrt{25} = \underline{\underline{5}}$$



and the angle relative to $x_1 - x_2$:

$$\tan^{-1}\left(\frac{r_2}{r_1}\right) = \tan^{-1}\left(-\frac{3}{4}\right) = \underline{\underline{-37°}}$$

<u>Step 3</u> Do the same for the rotated $\tilde{x}_1 - \tilde{x}_2$ system

$$|\tilde{r}_{1-2}| = \sqrt{(\tilde{r}_1)^2 + (\tilde{r}_2)^2}$$

$$= \sqrt{(4.893)^2 + (-1.026)^2}$$

$$= \sqrt{24.99} \quad \text{goes to} \quad \underline{\underline{5}}$$

<u>the same</u> ✓

and then the angle relative to $\tilde{x}_1 - \tilde{x}_2$:

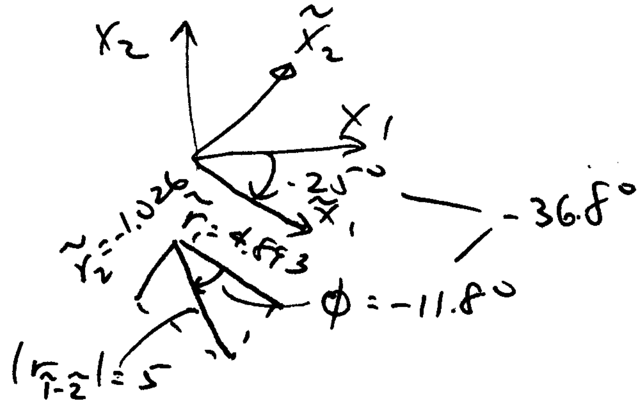
$$\tan^{-1}\left(\frac{\tilde{r}_2}{\tilde{r}_1}\right) = \tan^{-1}\left(\frac{-1.026}{4.893}\right)$$

$$= \tan^{-1}(-0.210)$$

$$= -11.8^\circ$$

Now include the rotation of -25° from $\tan^{-1}\left(\frac{r_2}{r_1}\right)$: $-11.8^\circ + (-25^\circ) = -\underline{\underline{36.8}}^\circ$

~ <u>same</u> as $\phi = 37^\circ$



$$\boxed{\text{Proven}}$$

<u>Q.E.</u>D.